# Measuring the Length of Period for the Long-Run Equilibrium in a Cointegration Relation

**Jae-Young Kim***

In economics the period of "long-run" often signifies the length of time within which transient fluctuations disappear, and a system comes back to an equilibrium state (or path). Among some interesting cases of long run analysis, the concept of cointegration is a relatively new concept of the long run equilibrium. This paper discusses how to determine the length of the long-run period for a cointegration relation. In an application to a consumption-income relation for three countries, U.S., Germany and Japan, we found that the length of the long-run period for the relation for these countries is about two to three years.

*Keywords*: Long run equilibrium, Cointegration, Consumption-income relation.

*JEL Classification*: C1, C22, C5

## I. Introduction

The concept of cointegration defined by Engle and Granger (1987) has become a useful concept for analyzing many linear dynamic systems in economics. For a set of variables the existence of a cointegration relation implies that there exists a long-run equilibrium that ties the series together. Thus, although disturbances to

*Associate Professor, School of Economics, Seoul National University, San 56-1 Shilim-dong Kwanak-gu, Seoul 151-742, Korea, (Tel) +82-2-880-6390, (E-mail) jykim017@snu.ac.kr. The author thanks the editor and anonymous referees for valuable comments. Financial supports from Seoul National University Foundation and a BK21 Program of Korea Research Foundation are greatly appreciated. Thanks also go to Joo-Hee Oh for assistance in editing the final draft of the paper in a word file.

individual variable(s) have permanent effects, they have only temporary effects on the system as a whole. When a shock is given to a system, the system deviates from the initial equilibrium. However, after a while, impacts of the shock are absorbed in the system, and the system eventually comes back to the initial equilibrium.

An interesting question we want to study in this paper is how long it takes for a system to return to the long-run equilibrium after a shock disturbs the system. A dynamic system seldom stays in an equilibrium, if any, even for a very short period of time because shocks are given to the system at each period of time and they survive for a while. An equilibrium error is usually an accumulation of current and past shocks to the system. Our study in this paper is to find how long it takes for shocks given at one period of time to disappear, that is, how long for shocks given at one period survive. We consider the consumption-spending model analyzed by Davidson, Hendry, Srba, and Yeo (1978) as an example of a cointegrated system. For data from three countries, Germany, Japan and U.S., we find that a shock to the system is shown to survive for nine, ten and twelve quarters, respectively, in the consumption-spending model. This implies that for the system of consumption-income relation it takes about two to three years to return to the state of long-run equilibrium after a shock disturbs the system.

Our discussion in this paper goes as follows. Section II discusses the methodology of our analysis. In section III we apply the method to the model of consumption-spending and discuss implications of our results. Mathematical proofs are provided in the appendix.

## II. Methodology

Let $x_t$ be an $n$-vector of I(1) variables. Suppose that there exists a cointegration relation among the variables. Then, there exists an $(n \times 1)$ vector $\alpha$ such that

$$\alpha' x_t = u_t \tag{1}$$

is a stationary process. A cointegration relation such as (1) is often interpreted as a long-run equilibrium relation among the variables.

That is, there exists a stable relation among the variables over a long period of time although there exist short-run fluctuations around the stable long-run relation. The short-run fluctuation $u_t$ can be interpreted as an equilibrium error to the system. Usually the error term $u_t$ contains shocks given in current and past periods.

Often the stationary error term $u_t$ is assumed to follow an AR process:[1]

$$u_t = a + \phi_1 u_{t-1} + \phi_2 u_{t-2} + \cdots + \phi_p u_{t-p} + \varepsilon_t \tag{2}$$

where $\varepsilon_t \sim i.i.d.(0, \sigma^2_\varepsilon)$ and $E(\varepsilon_t^4) < \infty$; all roots of

$$1 - \phi_1 z + \phi_2 z^2 + \cdots + \phi_p z^p = 0$$

lie outside the unit circle. The stationary error term $u_t$ can be written in MA($\infty$) representation as

$$u_t = \mu + \varepsilon_t + \Psi_1 \varepsilon_{t-1} + \Psi_2 \varepsilon_{t-2} + \cdots \tag{3}$$

This MA representation implies that $u_t$ is an accumulation of the current and past shocks $\varepsilon$'s. The coefficient $\Psi_s$ measures the level of impacts of $\varepsilon_{t-s}$ on $u_t$ for $s = 0, 1, \cdots, \infty$. More formally, the coefficient $\Psi_s$ has the interpretation

$$\Psi_s = \frac{\partial u_{t+s}}{\partial \varepsilon_t},$$

that is, $\Psi_s$ evaluates the consequences of a one unit increase in the shocks of the past s period on the current equilibrium error.

In practice we can get the estimate of $\Psi_s$ by converting the estimate of AR coefficient $\phi$'s. Such an estimate of $\Psi_s$, $\hat{\Psi}_s = \Psi_s(\hat{\phi})$, has the following sample property where $\hat{\phi}$ is the LS estimator of $\hat{\phi}$:

***Lemma* 1** Assume that $u_t$ and $\varepsilon_t$ satisfy the conditions above in (2) with all roots of the equation $1 - \phi_1 z + \phi_2 z^2 + \cdots + \phi_p z^p = 0$ lying outside the unit circle. Then, the estimator of $\Psi_s$, $\hat{\Psi}_s = \Psi_s(\hat{\phi})$, is such

[1]For example, in the augmented Dickey-Fuller test for cointegration we assume that the error term is a stationary AR process.

that

$$\sqrt{T}(\hat{\Psi}_s - \Psi_s) \xrightarrow{D} N(0, \sigma^2(\Psi_s)) \tag{4}$$

where $\sigma^2(\Psi_s) = G_s(\sigma^2_\varepsilon \otimes Q^{-1})G_s'$, for $Q = E(\nu_t \nu_t')$, $\nu_t' = (1 u_{t-1} u_{t-2} \cdots u_{t-p})$ and $\{G_s, s = 1, 2, \cdots\}$ is obtained by iterating on

$$G_s = (0 \Psi_{s-1} \Psi_{s-2} \cdots \Psi_{s-p}) + \phi_1 G_{s-1} + \phi_2 G_{s-2} + \cdots + \phi_p G_{s-p} \tag{5}$$

where $G_s = 0$ for $s \leq 0$, $\Psi_0 = 1$ and $\Psi_s = 0$ for $s < 0$.

Now, we are interested in the following hypothesis

$$H_0: \Psi_s = 0 \text{ for } s > \tau \text{ and } \Psi_s \neq 0 \text{ for } s \leq \tau, \tag{6}$$

which implies that the shock $\varepsilon$ survives for $\tau$ periods and after that it either disappears or is absorbed into the system.[2] We can decide that the hypothesis is true for a given $\tau$ if

$$\left| \frac{\sqrt{T}\hat{\Psi}_s}{\sqrt{s^2(\Psi_s)}} \right| \leq C_k \text{ for } s > \tau \text{ and } \left| \frac{\sqrt{T}\hat{\Psi}_s}{\sqrt{s^2(\Psi_s)}} \right| > C_k \text{ for } s \leq \tau. \tag{7}$$

for a given value $C_k$, where $s^2(\Psi_s)$ is the OLS estimate of $\sigma^2(\Psi_s)$. From (4) we know that $C_k$ can be asymptotically approximated by a critical value of the $t$ distribution.

In many cases of practice the disturbance $u_t$ is not observed. When $u_t$ is not observed, one may want to use an estimate of it, for example, $\hat{u}_t = \hat{\alpha}' x_t$ where $\hat{\alpha}$ is an estimate of $\alpha$. Statistical behavior of $\hat{u}_t$, however, is different from that of $u_t$ although $\hat{u}_t$ converges to $u_t$ as sample size gets larger.[3] Thus, the above asymptotic result may not be a good approximate for finite sample analysis when $\hat{u}_t$ is used in place of $u_t$. In such a case we may apply a re-sampling method such as the bootstrap method to get a

[2] The hypotheses in (6) can be generalized to $H_0: |\Psi_s| \leq \in$ for $s > \tau$ and $|\Psi_s| > \in$ for $s \leq \tau$ for a small $\in > 0$. In particular, it would be more appropriate to employ this generalized setup when one considers conversion of the AR coefficients to MA coefficients to get $\Psi_s$. Our hypotheses in (6) approximate this generalized setup with an infinitesimal value of $\in$. This idea came out of a comment from one of referees. I appreciate the referee for it.

[3] Note that $\hat{u}_t = u_t + (\hat{\alpha} - \alpha)' x_t$ and $\hat{\alpha} - \alpha = O_p(T^{-1})$, $x_t = O_p(t^{1/2})$ for a cointegrating vector $\alpha$ and a vector of $I(1)$ variables $x_t$ without drift.

critical value $C_k$.

## III. An Example: A Consumption-Income Relation

We now consider an example of a cointegration relation to find how long it takes for the system to return to its initial equilibrium after a shock disturbs the system. The example we consider in this section is a cointegration regression formed by two variables, consumption and income, denoted by $c$ and $y$, respectively:

$$c_t = \beta_0 + \beta_1 y_t + u_t \tag{8}$$

The regression (8) is often studied in the literature of cointegration, for example, Davidson, Hendry, Srba, and Yeo (1978).

We consider data from three countries, U.S., Japan, and Germany. Data from U.S. are quarterly time series on real personal consumption ($c$) and real personal disposable income ($y$) from 1947:1 through 1994:1. Data from the other countries are quarterly observations on real private consumption and real national income. By an augmented Dickey-Fuller test the unit root null is not rejected at 5% level for both $y$ and $c$ for data of all three countries.[4] Also, for both an augmented Dickey-Fuller test and Phillips and Ouliaris (1990) tests of cointegration we reject the null of no cointegration at 5% level for the consumption-income relation.

Tables 3.1(a)-(c) provide values of the statistic $t(s) = |\sqrt{T}\hat{\Psi}_s / \sqrt{s^2(\Psi_s)}|$ for $s = 1, 2, \cdots, 16$ for data of the three countries. Figures 3.1(a)-(c) exhibit the information in Tables 3.1(a)-(c) graphically. From Table 3.1(a) and Figure 3.1(a) we find that for U.S. data the period $\tau$ with $11 < \tau < 12$ satisfies the condition (7) with the 5% critical value 1.96. This result implies that for the U.S. case it takes about 12 quarters for the consumption-income relation to return to its initial equilibrium after a shock disturbs the system. Also, from Tables 3.1(b)-(c) and Figures 3.1(b)-(c) we find that the periods for the relation to return to its initial equilibrium are nine and ten quarters, respectively, for Germany and Japan.

[4]The results of tests for a unit root and cointegration are available from the author upon request.

**TABLE 3.1(a)**
*t*(*s*)-VALUES, U.S.

| $s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $t(s)$ | 8.29 | 6.58 | 5.56 | 5.53 | 4.83 | 4.04 | 3.43 | 2.97 |
| $s$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| $t(s)$ | 2.61 | 2.32 | 2.09 | 1.89 | 1.73 | 1.60 | 1.48 | 1.38 |

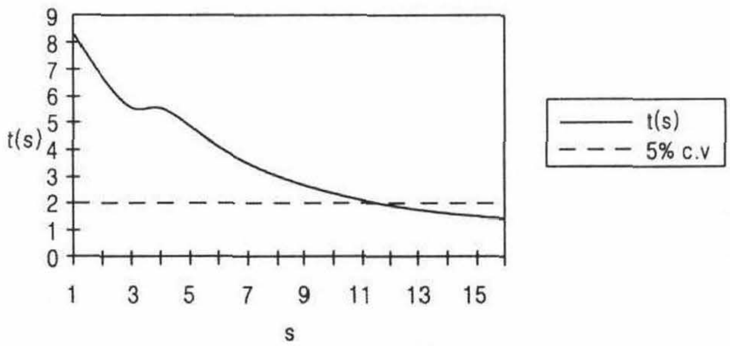


**FIGURE 3.1(a)**

**TABLE 3.1(b)**
*t*(*s*)-VALUES, GERMANY

| $s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $t(s)$ | 7.97 | 4.84 | 5.03 | 4.55 | 3.49 | 3.03 | 2.47 | 2.05 |
| $s$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| $t(s)$ | 1.80 | 1.56 | 1.37 | 1.24 | 1.12 | 1.02 | 0.94 | 0.87 |

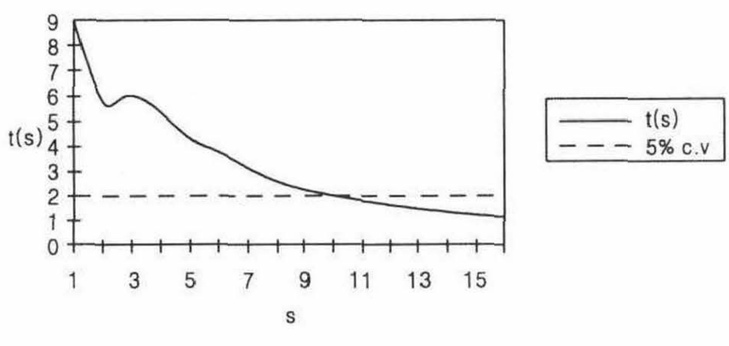


**FIGURE 3.1(b)**

TABLE 3.1(c)
$t(s)$-VALUES, JAPAN

| $s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $t(s)$ | 7.67 | 4.44 | 4.79 | 4.72 | 3.67 | 3.21 | 2.76 | 2.36 |
| $s$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| $t(s)$ | 2.10 | 1.87 | 1.68 | 1.53 | 1.40 | 1.29 | 1.20 | 1.12 |

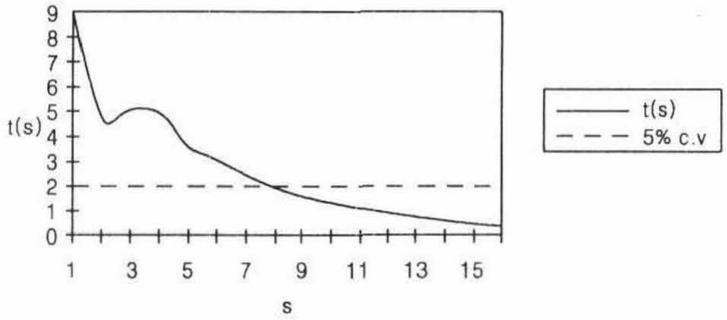


FIGURE 3.1(c)

It would be interesting to explore economic reasons why there exists such difference across countries in the length of the period for the long-run equilibrium.[5] This is an important subject to study. However, it is beyond the scope of this paper and is set aside for future research. The difference might have something to do with differences in economic fundamentals or economic policies across the countries.

## Appendix A: *Proof of Lemma 1:*

The following proof is based on several lemmas and results summarized in Hamilton (1994). Some of those lemmas or results are also available elsewhere, for example White (1984).

***Lemma A.1:*** Assume that $u_t$ satisfies conditions in Lemma 1. Let $\pi=(c, \phi_1, \cdots, \phi_p)$ and $\hat{\pi}_T$ be the OLS estimator of $\pi$ based on the sample of size $T$. Also, let $\hat{\sigma}^2_\varepsilon = T^{-1}\sum_{t=1}^T \hat{\varepsilon}_t^2$ where $\hat{\varepsilon}_t$ is the OLS

[5]A referee suggested to include this issue to better motivate the work of this paper. I appreciate the referee for it.

estimate of $\varepsilon_t$. Then, it is true that

(a) $T^{-1}\sum_{t=1}^{T}\nu_t\,\nu_t' \xrightarrow{p} Q$;

(b) $\hat{\pi}_r \xrightarrow{p} \pi$;

(c) $\hat{\sigma}^2_\varepsilon \xrightarrow{p} \sigma^2_\varepsilon$

(d) $\sqrt{T}(\hat{\pi}_r - \pi) \xrightarrow{D} N(0, \sigma^2_\varepsilon \otimes Q^{-1})$.

***Proof of Lemma A.1:***

(a) The condition of all roots of $1-\phi_1 z+\phi_2 z^2+\cdots+\phi_p z^p=0$ lying outside the unit circle ensures that the MA($\infty$) representation is absolutely summable. Then, it follows that $u_t$ is ergodic for the first moment and for the second moment. Then, the conclusion of (a) follows.
(b) It can be shown that $Y_t \equiv \nu_t\,\varepsilon_t$ satisfies conditions for the following law of large numbers for $L^1$-mixingales by Andrews (1988):

***Lemma A.2:*** Let $\{Y_t\}$ be an $L^1$-mixingale. If $\{Y_t\}$ is uniformly integrable and there exists a choice for $\{c_t\}$ such that $\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} c_t < \infty$, then $T^{-1}\sum_{t=1}^{T} Y_t \xrightarrow{p} 0$.
By Lemma A.2 the conclusion of (b) follows since $\hat{\pi} = \pi + (T^{-1}\sum_{t=1}^{T}\nu_t\,\nu_t')^{-1}(T^{-1}\sum_{t=1}^{T}\nu_t\,\varepsilon_t)$. (c) It follows from the conclusion of (*b*) and a law of large numbers.
(d) We will use the following result:

***Lemma A.3:*** (Corollary 5.25 of White (1984, p. 130)). Let $\{Y_t\}_{t=1}^{\infty}$ be a martingale difference sequence with $\bar{Y}_t = (1/T)\sum_{t=1}^{T} Y_t$. Suppose that (a) $E(Y_t^2) = \sigma_t^2 > 0$ with $(1/T)\sum_{t=1}^{T}\sigma_t^2 \to \sigma^2 > 0$ (b) $E|Y_t|^r < \infty$ for some $r > 2$ and all $t$, and (c) $(1/T)\sum_{t=1}^{T} Y_t^2 \to \sigma^2$. Then $\sqrt{T}\bar{Y}_T \to N(0, \sigma^2)$.
Notice that

$$\sqrt{T}(\hat{\pi} - \pi) = (T^{-1}\sum_{t=1}^{T}\nu_t\,\nu_t')^{-1}(T^{-1/2}\sum_{t=1}^{T}\nu_t\,\varepsilon_t).$$

Also, notice that $Y_t = \nu_t\,\varepsilon_t$ is a martingale difference sequence with finite fourth moments and $E(Y_t Y_t') = \sigma^2_\varepsilon \otimes Q$. It is not hard to show that

$$T^{-1}\sum_{t=1}^{T} Y_t Y_t' \xrightarrow{p} \sigma^2_\varepsilon \otimes Q.$$

Then, it follows from Lemma A.3 that

$$T^{-1/2}\sum_{t=1}^{T} Y_t \xrightarrow{D} N(0,(\sigma^2_\varepsilon \otimes Q)).$$

Then, since $T^{-1}\sum_{t=1}^{T} \nu_t \nu_t' \xrightarrow{p} Q$ as is shown in (a), the conclusion of (d) follows.
Now, as the final step we utilize the following result:

***Lemma A.4:*** (Corollary 5.25 of White (1984, p. 130)). Let $\{X_t\}$ be a sequence of random $(n \times 1)$ vectors such that $\sqrt{T}(X_T - c) \xrightarrow{D} X$ and let $g: R^n \rightarrow R^m$ have continuous first derivatives with $G$ denoting the $(m \times n)$ matrix of derivatives evaluated at $c$: $G = \partial g / \partial X'|_{x=c}$ Then, $\sqrt{T}(g(X_T) - g(c)) \xrightarrow{D} GX$.
Now, let $X_T = \hat{\pi}_T$, $c = \pi$, and $X \sim N(0, \sigma^2_\varepsilon \otimes Q^{-1})$. Also, let $g_s = \Psi_s$ so that $G_s = \partial \Psi_s(\pi) / \partial \pi'$. Then we have

$$\sqrt{T}(\hat{\Psi}_s - \Psi_s) \xrightarrow{D} N(0, G_s(\sigma^2_\varepsilon \otimes Q^{-1}) G_s').$$

The equivalence of $G_s = \partial \Psi_s(\pi) / \partial \pi'$ and $G_s$ in (5) can be shown by a long but straightforward algebra. See Hamilton (1994, pp. 344-8).

(*Received 22 November 2003; Revised 20 February 2004*)

## References

Davidson, James E. H., Hendry, David F., Srba, Frank, and Yeo, Stephen. "Econometric Modelling of the Aggregate Time-Series Relationship between Consumer's Expenditure and Income in the United Kingdom." *The Economic Journal* 88 (No. 352 1978): 661-92.

Engle, Robert F., and Granger, C. W. J. "Cointegration and Error Correction: Representation, Estimation, and Testing." *Econometrica* 55 (No. 2 1987): 251-76.

Hamilton, J. *Time Series Analysis*. Princeton, NJ: Princeton U. Press, 1994.

Phillips, P. C. B., and Ouliaris, "Asymptotic Properties of Residual Based Tests for Cointegration." *Econometrica* 58 (No. 1 1990):

165-93.

White, H. *Asymptotics Theory for Econometricians*. Academic Press 1984.